NUMERICAL SCHEMES AND STABILITY ANALYSIS FOR FRACTIONAL MODELS OF TRANSIENT UNI- AND BIDIMENSIONAL FLOW IN PETROLEUM RESERVOIRS

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ABSTRACT

Objective: To propose a modeling for the flow in oil reservoirs using Caputo’s definition of fractional derivative applied to the spatial coordinate of the problem. Furthermore, delimit a region of numerical stability for the explicit method of solving the model equations.

Methodology: Starting from the material balance in a differential control volume and the generalized Darcy’s law, and coupling it with an equation of state, models describing flow in porous media were derived. A numerical solution scheme using an explicit discretization was proposed. For this purpose, the L1 method for approximating spatial fractional derivatives and finite difference approximation for temporal derivatives were employed.

Results and Discussion: Through the Von Neumann stability analysis, it was observed that the explicit solution schemes for the models are conditionally stable and dependent on the order of the fractional derivative. Additionally, numerical stability regions were constructed for one and two-dimensional transient schemes, revealing that the one-dimensional scheme is stable for \( M \leq 0.5 \), and the two-dimensional scheme is stable for \( M \leq 0.25 \) for all derivative orders between 0 and 1.

Implications of the research: Historically, flow models in petroleum reservoirs have been based on the application of Darcy’s law; however, its use is limited due to some constraints. Recently, fractional calculus has played a significant role in generalizing and obtaining more accurate models in various application areas. In this context, Darcy’s law has been generalized, allowing for the derivation of fractional models that are more suitable for describing flow in porous media. These models, in general, are solved using a numerical method, and it is crucial to understand the stability region of the method for their resolution.

Keywords: Stability Analysis, Flow in Porous Media, Fractional Calculus.

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ESQUEMAS NUMÉRICOS E ANÁLISE DE ESTABILIDADE PARA MODELOS FRACIONAIS DE FLUXO TRANSITÓRIO UNI E BIDIMENSIONAL EM RESERVATÓRIOS DE PETRÓLEO

RESUMO

Objetivo: propor uma modelagem para o escoamento em reservatórios de petróleo usando a definição de derivada fracionária de Caputo aplicada a coordenada espacial do problema. Além disso, delimitar uma região de estabilidade numérica para o método explícito de solução das equações do modelo.

Metodologia: partindo do balanço material em um volume diferencial de controle e da lei de Darcy generalizada, e acoplando-se uma equação de estado, foram obtidos modelos que descrevem o escoamento em meios porosos. Um esquema numérico de solução utilizando uma discretização explícita foi proposto. Para isso, utilizou-se o método L1 de aproximação das derivadas fracionárias espaciais e a aproximação por diferenças finitas para as derivadas temporais.

Resultados e discussão: observou-se, através da análise de estabilidade de Von Neumann, que os esquemas explícitos de solução dos modelos são condicionalmente estáveis e dependentes da ordem da derivada fracionária. Além disso, construiu-se as regiões de estabilidade numérica para os esquemas uni e bidimensional transitente, mostrando que o esquema unidimensional é estável para \( M \leq 0,5 \) e que o esquema bidimensional é estável para \( M_2 \leq 0,25 \) para toda ordem da derivada entre 0 e 1.

Implicações da pesquisa: historicamente, os modelos de escoamento em reservatórios de petróleo se baseiam na aplicação da lei de Darcy, todavia, seu uso é restrito devido a algumas limitações. Recentemente, o cálculo fracionário vem desempenhando um papel importante na generalização e obtenção de modelos mais precisos em diversas áreas de aplicação. Nesse sentido, a lei de Darcy foi generalizada possibilitando a obtenção de modelos fracionários mais adequados à descrição do escoamento em meios porosos. Esses modelos, em geral, são resolvidos através de um método numérico, sendo fundamental o conhecimento da região de estabilidade do método.


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1 INTRODUCTION

In simplified terms, an oil reservoir consists of a permeable, porous, or fractured rock containing hydrocarbons and water. This rock serves as the storage space for hydrocarbons and plays a crucial role in regulating fluid flow within the reservoir (Obembe et al., 2017). The hydrocarbon systems found in reservoirs are complex mixtures of organic compounds, typically comprising oil and gas, along with water, exhibiting multiphase behavior over wide ranges of temperature and pressure (Ahmed, 2010).

Due to this complexity, mathematical modeling of hydrocarbon flow is essential for evaluating reservoir performance and yield, as well as enabling the optimization of production and flow parameters under various conditions. These models are based on physical and mathematical principles to describe the behavior of these systems, such as material balance, flow and continuity equations, equation of state, and empirical correlations. (Ertekin et al., 2001).

Historically, flow models in porous media and, consequently, in oil reservoirs have been based on the application of Darcy's law, which was experimentally derived by correlating the flow velocity of fluid through a certain amount of porous material to its pressure gradient. However, its use is limited due to some simplifications, such as assuming homogeneous, single-
phase, Newtonian fluid flow, laminar flow, and medium properties independent of pressure and temperature (Awotunde et al., 2016).

Recently, fractional calculus has been playing a significant role in generalizing and obtaining more accurate models in various fields of application. In this context, Darcy’s law has been generalized, enabling the derivation of fractional models that are more suitable for describing flow in porous media. (Ren and Guo, 2015).

The theory of fractional calculus originated in the late 17th and early 18th centuries, remaining for a long time as a field of pure mathematics without practical applications. However, this scenario has changed, and in recent years, many authors have demonstrated its enormous capacity as a tool for describing and solving practical problems (Battaglia et al., 2001; Vinnett et al., 2015; Yang et al., 2016; Ionescu et al., 2017; Jaques et al., 2017; Obembe et al., 2017; Xu and Jiang, 2017; Zhang et al., 2017; Chen et al., 2010; Chang et al., 2019).

As a consequence of this increased interest, new approaches have been developed, and different definitions of fractional derivatives and integrals are available. For the purpose of this work, the definition proposed by Caputo (Caputo, 1967), was chosen, as given by Equation (1), while Equation (2) presents the definition of fractional integral.

\[
D_{\alpha}^{a}f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{a}^{t} \frac{f^{n}(\varepsilon) d\varepsilon}{(t - \varepsilon)^{\alpha + 1 - n}}
\]

\[
I_{\alpha}^{a}f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - \varepsilon)^{\alpha - 1} f(\varepsilon) d\varepsilon
\]

In this work, we present the modeling of one-dimensional and two-dimensional transient flow using fractional derivatives applied to the spatial coordinate of the model. Additionally, an explicit solution algorithm based on the L1 approximation (Li and Zeng, 2015) of the Caputo derivatives was proposed, and its numerical stability region was studied using the Von Neumann method (Crank and Nicolson, 1996).

2 THEORETICAL FRAMEWORK

2.1 Modeling

Consider a differential control volume (dv) according to the scheme shown in Figure 1. For this volume, it is assumed that its faces are permeable to the mass flow perpendicular to its plane, as well as the possibility of adding or removing mass from the volume control via external source (well). Therefore, the material balance for this control volume is given by Equation 3.

\[
- \frac{\partial(\rho u_x A_x)}{\partial x} \Delta x - \frac{\partial(\rho u_y A_y)}{\partial y} \Delta y - \frac{\partial(\rho u_z A_z)}{\partial z} \Delta z + q_m = V_b \frac{\partial(\phi \rho)}{\partial t}
\]

Where:

- \(\rho\) is the fluid density,
- \(u_i\) is the flow velocity, perpendicular to the area \(A_i\),
- \(\Delta x, \Delta y\) and \(\Delta z\) are the dimensions of the control volume,
- \(q_m\) is the mass flow via external source (well),
- \(V_b\) is the volume and \(\phi\) is the porosity.
Figure 1 – Schematic representation of the control volume
Source: authors (2023)

Equation 3 represents the differential form of the transport equation. However, to use it in flow problems in oil reservoirs, it is necessary to couple an equation of state, as well as the generalized Darcy equation. As suggested by Ertekin et al., 2001, the equation of state can be coupled to the flow model via volume formation factor (B), Equation (4).

\[
B = \frac{V}{V_{sc}} = \frac{\rho_{sc}}{\rho} \tag{4}
\]

The flow velocity, on the other hand, will be described by the generalized Darcy equation, Equations (5), (6), and (7). For this work, it is of particular interest to write the Darcy equation using the Caputo fractional derivative definition with respect to the spatial coordinate.

\[
u_x = -\beta_c k_x \frac{\partial^\alpha p}{\mu \partial x^\alpha} \tag{5}
\]

\[
u_y = -\beta_c k_y \frac{\partial^\alpha p}{\mu \partial y^\alpha} \tag{6}
\]

\[
u_z = -\beta_c k_z \frac{\partial^\alpha p}{\mu \partial z^\alpha} \tag{7}
\]

Where:

- \(\beta_c\) is a proportionality constant between the English and metric unit systems and has a value of 1.127.
- \(k_i\) represents the reservoir pseudopermeability in the direction \(i\).
\( \mu \) is the phase viscosity and \( \frac{\partial^\alpha P}{\partial x^\alpha} \) is the Caputo operator of the fractional derivative of pressure with respect to the flow direction.

By substituting equations (5) to (7) into the material balance, equation (3), the fractional flow model in petroleum reservoirs, Equation (8), is obtained. Equation (8) describes the flow in a reservoir in a general form. Hence, from this equation, other cases can be derived.

\[
\frac{\partial}{\partial x} \left( \frac{A_x \beta_c k_x \frac{\partial^\alpha P}{\partial x^\alpha}}{B \mu} \right) \Delta x + \frac{\partial}{\partial y} \left( \frac{A_y \beta_c k_y \frac{\partial^\alpha P}{\partial y^\alpha}}{B \mu} \right) \Delta y + \frac{\partial}{\partial z} \left( \frac{A_z \beta_c k_z \frac{\partial^\alpha P}{\partial z^\alpha}}{B \mu} \right) \Delta z + q_{sc} = V_b \frac{\partial}{\partial t} \left( \phi \right)
\]

(8)

2.1 Incompressible Fluid Flow

Special importance is given to the case where the fluid can be considered incompressible, meaning its density is not a function of pressure, and therefore, \( B = 1 \) making the model independent of time, as shown in Equation (9).

\[
\frac{\partial}{\partial x} \left( \frac{A_x \beta_c k_x \frac{\partial^\alpha P}{\partial x^\alpha}}{\mu} \right) \Delta x + \frac{\partial}{\partial y} \left( \frac{A_y \beta_c k_y \frac{\partial^\alpha P}{\partial y^\alpha}}{\mu} \right) \Delta y + \frac{\partial}{\partial z} \left( \frac{A_z \beta_c k_z \frac{\partial^\alpha P}{\partial z^\alpha}}{\mu} \right) \Delta z + q_{sc} = 0
\]

(9)

This is a very important result that tells us that for an incompressible fluid, the pressure profile along the reservoir is established instantaneously, immediately after the start of the operation. Furthermore, another point to consider is that Equation (9) also describes steady-state flow. Therefore, an incompressible flow problem with boundary conditions independent of time has a numerically identical solution to the steady-state flow problem (Ertekin et al., 2001).

Other simplifications can be made, depending on the specific case under study. For the flow of an incompressible fluid with constant viscosity, Equation (10) can be applied.

\[
\frac{\partial}{\partial x} \left( A_x \beta_c k_x \frac{\partial^\alpha P}{\partial x^\alpha} \right) \Delta x + \frac{\partial}{\partial y} \left( A_y \beta_c k_y \frac{\partial^\alpha P}{\partial y^\alpha} \right) \Delta y + \frac{\partial}{\partial z} \left( A_z \beta_c k_z \frac{\partial^\alpha P}{\partial z^\alpha} \right) \Delta z + \mu q_{sc} = 0
\]

(10)

If the reservoir is anisotropic \( (k_x \neq k_y \neq k_z) \), but homogeneous, the pseudopermeabilities \( (k_x, k_y \text{ e } k_z) \) will remain constant throughout the entire reservoir, and thus the balance simplifies according to Equation (11).

\[
k_x \frac{\partial}{\partial x} \left( \frac{\partial^\alpha P}{\partial x^\alpha} \right) + k_y \frac{\partial}{\partial y} \left( \frac{\partial^\alpha P}{\partial y^\alpha} \right) + k_z \frac{\partial}{\partial z} \left( \frac{\partial^\alpha P}{\partial z^\alpha} \right) + \frac{\mu q_{sc}}{\beta_c V_b} = 0
\]

(11)

For an isotropic medium where the permeabilities are equal and constant \( (k_x = k_y = k_z = k) \), the model simplifies even further, resulting in Equation (12).

\[
\frac{\partial}{\partial x} \left( \frac{\partial^\alpha P}{\partial x^\alpha} \right) + \frac{\partial}{\partial y} \left( \frac{\partial^\alpha P}{\partial y^\alpha} \right) + \frac{\partial}{\partial z} \left( \frac{\partial^\alpha P}{\partial z^\alpha} \right) + \frac{\mu q_{sc}}{\beta_c k V_b} = 0
\]

(12)
Finally, when dealing with the flow of an incompressible fluid with constant viscosity, in an isotropic and homogeneous medium, and with no external source in the control volume ($q_{sc} = 0$), the model reduces to Equation (13).

$$\frac{∂}{∂x} \left( \frac{∂^{α}P}{∂x^{α}} \right) + \frac{∂}{∂y} \left( \frac{∂^{α}P}{∂y^{α}} \right) + \frac{∂}{∂z} \left( \frac{∂^{α}P}{∂z^{α}} \right) = 0$$ \hspace{1cm} (13)

2.2 Slightly Compressible Fluid Flow

For the case where the fluid is slightly compressible ($B ≠ 1$), meaning that the fluid density is weakly influenced by pressure, some changes must be made to the model. Initially, it is assumed that the fluid compressibility is small ($10^{-6} \text{psi} < c < 10^{-5} \text{psi}$) and remains constant within the pressure range of interest, and that the volume formation factor can be expressed by equation (14).

$$B = \frac{B^0}{[1 + c(P - P^0)]}$$ \hspace{1cm} (14)

Where $c$ is the fluid compressibility, given by equation (15).

$$c = \frac{1}{\rho} \frac{∂\rho}{∂P} \bigg|_T$$ \hspace{1cm} (15)

Therefore, by substituting Equation (14) into Equation (8) and manipulating it, we obtain the fractional equation that describes the flow of a slightly compressible fluid in a porous medium, Equation (16).

$$\frac{∂}{∂x} \left( \frac{A_x β_c k_x}{μB^0} \frac{∂^{α}P}{∂x^{α}} \right) Δx + \frac{∂}{∂y} \left( \frac{A_y β_c k_y}{μB^0} \frac{∂^{α}P}{∂y^{α}} \right) Δy + \frac{∂}{∂z} \left( \frac{A_z β_c k_z}{μB^0} \frac{∂^{α}P}{∂z^{α}} \right) Δz + q_{sc} = \frac{V_bφ_c}{a_c B^0} \frac{∂P}{∂t}$$ \hspace{1cm} (16)

3 METHODOLOGY

3.1 Explicit Discretization

The fractional models obtained are partial differential equations of fractional order, which are often difficult or even impossible to solve analytically. For such cases, numerical solutions of the problem are used. The first step to obtain the numerical solution is to discretize the domain. We consider the one-dimensional transient flow of a slightly compressible fluid, given by Equation (17), and a centered blocks grid with $n_x$ elements overlaid on a reservoir in Cartesian coordinates, as shown in Figure 2.

$$\frac{∂}{∂x} \left( \frac{A_x β_c k_x}{μB} \frac{∂^{α}P}{∂x^{α}} \right) Δx + q_{sc} = \frac{V_bφ_c}{a_c B^0} \frac{∂P}{∂t}$$ \hspace{1cm} (17)
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Figure 2 - Centered blocks grid with \( n_x \) elements of dimension \( \Delta x_i \)

Source: authors (2023)

These blocks have defined dimensions \( \Delta x_i \) which may not necessarily be equal, but must satisfy Equation (18).

\[
\sum_{i=1}^{n_x} \Delta x_i = L_x (18)
\]

Once the grid is defined, the points where the pressures will be effectively calculated are placed at the center of each block. Therefore, the boundaries of the i-th block are represented by \( x_i \pm 1/2 \) and its center is represented by \( x_i \).

By applying numerical approximations, using finite differences for spatial and temporal derivatives, and taking the time \( t_n \) as the base time of the solution, Equation (19) is obtained.

\[
\left[ \frac{(A_x \beta_c k_x)^n}{\mu B} \left( \frac{\partial^\alpha P}{\partial x^\alpha} \right)_{i+1/2}^{n} - \frac{(A_x \beta_c k_x)^n}{\mu B} \left( \frac{\partial^\alpha P}{\partial x^\alpha} \right)_{i-1/2}^{n} \right] + q_{isc}^n = \frac{V_b \phi_c}{a_c B_0} \left( \frac{p_i^{n+1} - p_i^n}{\Delta t} \right) (19)
\]

The fractional derivatives at the boundaries of the blocks were discretized using the L1 approximation, given by Equations (20) and (21), as suggested by Li and Zeng, 2015; Awotunde et al., 2016b; Zhuang and Liu, 2006; Karatay et al., 2011; Murio, 2008)

\[
\frac{\partial^\alpha P}{\partial x^\alpha} \bigg|_{i+1/2}^{n} = \sigma \sum_{k=0}^{i} b_k [p_{i+1-k}^n - p_i^{n}] (20)
\]

\[
\frac{\partial^\alpha P}{\partial x^\alpha} \bigg|_{i-1/2}^{n} = \sigma \sum_{k=0}^{i-1} b_k [p_{i-k}^n - p_{i-1-k}^n] (21)
\]

\[
\sigma = \frac{1}{\Delta x^\alpha \Gamma(2 - \alpha)}
\]

\[
b_k = (k + 1)^{1-\alpha} - k^{1-\alpha}
\]

By defining the reservoir transmissibility in the flow direction, Equation (22), and substituting it into Equation (19) and rearranging, we obtain the explicit solution scheme, Equation (23).
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\[ T^n_{x\pm\frac{1}{2}} = \left( \frac{A_x\beta c k_x}{\mu B \Delta x} \right)_{i\pm\frac{1}{2}} \]  \hspace{1cm} (22)

\[ p_{i+1}^n = p_i^n + \Lambda_i^n \left[ \sum_{k=0}^{n} b_{k}[p_i^n - p_{i-k}^n] - \sum_{k=0}^{i-1} b_{k}[p_i^n - p_{i-k-1}^n] + q_{isc}^n \right] \]  \hspace{1cm} (23)

\[ \Lambda_i^n = \left( \frac{V_b \phi c}{a_c B^0 \Delta t} \right)^n_i \]

To complete the solution algorithm, the formulation and application of the initial and boundary conditions of the problem are required. For one-dimensional flow, it is reasonable to assume that the reservoir depth is relatively small, meaning that the pressure gradient between the top and bottom of the reservoir is negligible. Therefore, the initial pressure condition in the reservoir is defined as constant and equal to a certain pressure \( P_0 \), as given by Equation (24).

\[ P(x, 0) = P_0 \]  \hspace{1cm} (24)

For the boundary conditions, there are at least two commonly used ways to express them. The reservoir boundaries can be influenced by the surroundings and, therefore, maintained at a defined pressure, or the reservoir can be bounded by impermeable rocks, resulting in zero flow across its boundaries. Initially, we consider the isolated reservoir, according to Equations (25) and (26).

\[ \frac{\partial P(x, t)}{\partial x} \bigg|_{x=0} = 0 \]  \hspace{1cm} (25)

\[ \frac{\partial P(x, t)}{\partial x} \bigg|_{x=L} = 0 \]  \hspace{1cm} (26)

The same procedure can be applied to the problem of transient, two-dimensional, and slightly compressible fluid flow, as described by Equation (27).

\[ \frac{\partial}{\partial x} \left( \frac{A_x\beta c k_x}{\mu B^0} \frac{\partial^\alpha P}{\partial x^\alpha} \right) \Delta x + \frac{\partial}{\partial y} \left( \frac{A_y\beta c k_y}{\mu B^0} \frac{\partial^\alpha P}{\partial y^\alpha} \right) \Delta y + q_{sc} = \frac{V_b \phi c}{a_c B^0} \frac{\partial P}{\partial t} \]  \hspace{1cm} (27)

For a grid with \( n_x \times n_y \) blocks, overlaid on a reservoir in rectangular coordinates, with defined dimensions \( \Delta x_i \) e \( \Delta y_i \) that satisfy Equations (28) and (29).

\[ \sum_{i=1}^{n_x} \Delta x_i = L_x \]  \hspace{1cm} (28)

\[ \sum_{i=1}^{n_y} \Delta x_i = L_y \]  \hspace{1cm} (29)
Defining the transmissibilities in the $x$ and $y$ directions according to Equations (30) and (31).

\[
T^n_{x,i_{\pm 1/2},j} = \left( \frac{A_x \beta_c k_x}{\mu B \Delta x} \right)^n_{i_{\pm 1/2},j} 
\]

\[
T^n_{y,i,j_{\pm 1/2}} = \left( \frac{A_y \beta_c k_y}{\mu B \Delta y} \right)^n_{i,j_{\pm 1/2}} 
\]

Assuming that the fractional order derivatives can be approximated using the L1 method. The explicit solution scheme for Equation (27) is given by Equation (32)

\[
P_{i,j}^{n+1} = P_{i,j}^n + \Lambda_{i,j}^n \left[ T^n_{x,i_{\pm 1/2},j} \sum_{k=0}^i b_k [P^n_{i+1-k,j} - P^n_{i-k,j}] 
- T^n_{x,i_{-1/2},j} \sum_{k=0}^{i-1} b_k [P^n_{i-k,j} - P^n_{i-1-k,j}] 
+ T^n_{y,i,j_{\pm 1/2}} \sum_{k=0}^{j-1} b_k [P^n_{i,j+1-k} - P^n_{i,j-k}] 
- T^n_{y,i,j_{-1/2}} \sum_{k=0}^{j-1} b_k [P^n_{i,j-k} - P^n_{i,j-1-k}] + q^n_{isc} \right] 
\]

\[
\Lambda_{i,j}^n = \left( \frac{V_b \phi c}{a_c B^0 \Delta t} \right)^n_{i,j} 
\]

\[
\sigma_x = \frac{1}{\Delta x^\alpha - 1 \Gamma(2 - \alpha)} 
\]

\[
\sigma_y = \frac{1}{\Delta y^\alpha - 1 \Gamma(2 - \alpha)} 
\]

The initial condition of pressure in the reservoir is defined as constant and equal to a specific pressure $P_0$, according to Equation (33).

\[
P(x, y, 0) = P_0 
\]

The reservoir is considered to be located between impermeable rocks, and consequently, the flow through its boundaries is zero. Thus, the boundary conditions of the problem are given by Equations (34) to (37).

\[
\frac{\partial P(x, y, t)}{\partial x} \bigg|_{x=0} = 0 
\]

\[
\frac{\partial P(x, y, t)}{\partial x} \bigg|_{x=L} = 0 
\]
$$\frac{\partial P(x, y, t)}{\partial y} \bigg|_{y=0} = 0 \quad (36)$$

$$\frac{\partial P(x, y, t)}{\partial y} \bigg|_{x=L} = 0 \quad (37)$$

### 4 RESULTS AND DISCUSSION

#### 4.1 Stability Analysis

Indeed, it is known that explicit discretization can lead to unstable algorithm formulations for obtaining numerical solutions (Crank and Nicolson, 1996; Awotunde et al., 2016b; Obembe et al., 2017b). In other words, not every grid is suitable for solving the problem. In general, explicit methods are conditionally stable, and understanding the stability region of the solution algorithm is essential.

For the determination of the stability criterion, the Von Neumann stability analysis method is applied to the transient one-dimensional flow problem. However, this method has some limitations, and to obtain a stability region, it is necessary to make some simplifications in the model.

Therefore, it is assumed that the transmissibilities are equal on both sides of the analyzed control volume, as given in Equation (38). Additionally, it is considered that there is no production/injection well in the reservoir.

$$T^n_{x_{i+\frac{1}{2}}} = \left(\frac{A_x \beta_c k_x}{\mu B \Delta x}\right)^n_{i+\frac{1}{2}} = T^n_x \quad (38)$$

Assuming that the initial condition, Equation (24), and the boundary conditions, Equations (25) and (26), also apply to this problem, the following solution algorithm is obtained.

For $i = 1$

$$P^n_{i+1} = P^n_i + M[b_0(P^n_2 - P^n_1)] \quad (39)$$

For $i = 2 \ldots N_x - 1$

$$P^n_{i+1} = P^n_i + M \left[ b_0(P^n_{i+1} - P^n_i) + \sum_{k=1}^{i-1} (b_k - b_{k-1})(P^n_{k+1} - P^n_k) \right] \quad (40)$$

For $i = N_x$

$$P^n_{i+1} = P^n_i + M \left[ \sum_{k=1}^{i-1} (b_k - b_{k-1})(P^n_{k+1} - P^n_k) \right] \quad (41)$$
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\[ M = \frac{T_x \sigma}{\Lambda} \]  \hspace{1cm} (42)

\[ \Lambda = \left( \frac{V_b \phi c}{\alpha c B^0 \Delta t} \right) \]  \hspace{1cm} (43)

\[ \sigma = \frac{1}{\Delta x^{\alpha-1} \Gamma(2-\alpha)} \]  \hspace{1cm} (44)

\[ b_k = (k + 1)^{1-\alpha} - k^{1-\alpha} \]  \hspace{1cm} (45)

The Von Neumann method considers that the error between the numerical solution and the analytical solution can be expressed through a Fourier series, and that this error can be decomposed into a temporal component and a spatial component, which should satisfy the discrete equation. Therefore, we can write Equations (46) to (49), where \( j \) is the imaginary unit defined as \( j = \sqrt{-1} \).

\[ P_i^{n+1} = \varepsilon^{n+1} e^{j(\omega x)} \]  \hspace{1cm} (46)

\[ P_i^n = \varepsilon^n e^{j(\omega x)} \]  \hspace{1cm} (47)

\[ P_{i+1}^{n+1} = \varepsilon^n e^{j[\omega(x+\Delta x)]} \]  \hspace{1cm} (48)

\[ P_{i-1}^{n+1} = \varepsilon^n e^{j[\omega(x-\Delta x)]} \]  \hspace{1cm} (49)

By substituting Equations (46) to (49) for \( i = 1 \) in the solution algorithm, Equation (39) becomes Equation (50).

\[ \varepsilon^{n+1} e^{j(\omega x)} = \varepsilon^n e^{j(\omega x)} + M[\varepsilon^n e^{j[\omega(x+\Delta x)]} - \varepsilon^n e^{j(\omega x)}] \]  \hspace{1cm} (50)

Stable solution schemes are capable of controlling the growth of error, and for this to hold true, the relationship between the error in the next temporal iteration \((n + 1)\) must be less than or equal to the error in the current temporal iteration \((n)\). Therefore, the error amplification factor \((\Lambda)\) must always be less than or equal to one, as shown in Equation (51).

\[ \lambda = \frac{|\varepsilon^{n+1}|}{|\varepsilon^n|} \leq 1 \]  \hspace{1cm} (51)

Dividing Equation (50) by \( \varepsilon^n e^{j(\omega x)} \), rearranging, and applying the Euler's formula, we arrive at Equation (52).

\[ \frac{|\varepsilon^{n+1}|}{|\varepsilon^n|} = |1 + M[\cos(\omega \Delta x) + jsen(\omega \Delta x) - 1]| \leq 1 \]  \hspace{1cm} (52)

Through the analysis of Equation (52), and recalling that \(|z| = \sqrt{[Re(z)]^2 + [Im(z)]^2}\) it is concluded that the numerical scheme will be stable at the point \(i=1\) if Equation (53) holds.
\[ [1 + M(\cos(\omega \Delta x) - 1)]^2 + [M \sin(\omega \Delta x)]^2 \leq 1 \]  

(53)

Therefore, \( M \) must satisfy the inequality (53) for the proposed scheme to be numerically stable at the point \( i = 1 \). It should be noted that the parameter \( M \) contains the proposed mesh for the discretization of the model. Analyzing the inequality (53), it is known that the cosine function is bounded by \(-1 \leq \cos(x) \leq 1\). By analyzing the limits of the cosine function, the relationship given by Equation (54) is obtained.

\[ 4M(M - 1) \leq 0 \]  

(54)

In other words, knowing that the parameter \( M \) is always positive, the inequality given in (54) is satisfied when:

\[ 0 < M \leq 1 \]  

(55)

Thus, it can be concluded that the proposed scheme will be conditionally stable at \( i = 1 \). This result is important as it confirms the conditionally stable nature of explicit schemes. However, it does not provide information about the stability at subsequent points in the grid. Therefore, we repeat the procedure considering \( i = 2 \). Applying Equations (46) to (49) to the numerical scheme equation for \( i = 2 \), Equation (40), we arrive at Equation (56).

\[ \varepsilon^{n+1} e^{j(\omega x)} = \varepsilon^n e^{j(\omega x)} + M \{ e^n e^{j[(\omega x + \Delta x)]} - e^n e^{j(\omega x)} + (b_1 - b_0)(e^n e^{j(\omega x)} - e^n e^{j(\omega (x - \Delta x))}) \} \]  

(56)

By repeating the procedure outlined for \( i=1 \), we arrive at the expression given by Equation (57) for the error amplification factor.

\[ \left| \frac{\varepsilon^{n+1}}{\varepsilon^n} \right| = |1 + M[(\cos(\omega \Delta x) - 1)(2 - b_1) + jb_1 \sin(\omega \Delta x)]| \leq 1 \]  

(57)

Analogously, the numerical scheme will be stable at \( i = 2 \) when the inequality (58) is satisfied.

\[ M^2[\cos(\omega \Delta x) - 1]^2(2 - b_1)^2 + 2M[\cos(\omega \Delta x) - 1](2 - b_1) + b_1^2[1 - \cos^2(\omega \Delta x)] \leq 0 \]  

(58)

Considering that \( M \) is always positive, the inequality (58) is satisfied within the interval given by equation (59).

\[ 0 < M \leq \frac{1}{2 - b_1} \]  

(59)

This result shows that for the second block of the grid, the explicit scheme is also partially stable. However, a more in-depth analysis of the interval expressed by Equation (59) reveals that the stability of the numerical scheme depends on the order of the derivative since \( b_k \) is a function of the derivative order \( (\alpha) \), Equation (21).
Considering that the interval for the parameter $M$ that guarantees the stability of the explicit scheme is a function of the derivative order, we proceed to perform numerical experiments aiming to determine the stability region of the explicit scheme.

Initially, it was observed that the explicit scheme is unstable for all $\alpha$ between 0 and 1 when $M=1.5$. Starting from this point, for the same spatial mesh, the temporal mesh was refined until the explicit scheme became stable. The results obtained from this numerical experiment are compiled in the graph shown in Figure (3). It is noted that, with some confidence, for $M \leq 0.5$ the explicit scheme is stable regardless of the derivative order.

![Figure 3 - Stability region for the transient one-dimensional explicit scheme.](image)

**Source:** authors (2023)

Motivated by the results obtained for the transient one-dimensional explicit scheme, the same stability analysis procedure was applied to the transient two-dimensional explicit scheme. Applying the initial and boundary conditions, the following solution algorithm is obtained:

For $i = 1, j = 1$

$$p_{1,1}^{n+1} = p_{1,1}^n + M\{b_0(P_{2,1} - P_{1,1}) + b_0(P_{1,2} - P_{1,1})\}$$

(60)

For $i = 2 \ldots N_x - 1, j = 1$

$$p_{i,j}^{n+1} = p_{i,j}^n + M\{b_0(P_{i+1,j} - P_{i,j}) + b_0(P_{i-1,j} - P_{i,j})\}$$

(61)
\[ P_{i,j}^{n+1} = P_{i,j}^n + M \left\{ b_0 (P_{i+1,j}^n - P_{i,j}^n) \right\} \\
\quad \quad \quad \quad \quad + \sum_{k=1}^{i-1} (b_k - b_{k-1}) (P_{k+1,j}^n - P_{k,j}^n) + b_0 \left( P_{i,j+1}^n - P_{i,j}^n \right) \]

For \( i = 1, j = 2 \ldots N_y - 1 \)

\[ P_{i,j}^{n+1} = P_{i,j}^n + M \left\{ b_0 (P_{i+1,j}^n - P_{i,j}^n) + b_0 (P_{i,j+1}^n - P_{i,j}^n) \right\} \]
\[ + \sum_{k=1}^{j-1} (b_k - b_{k-1}) (P_{k+1,j}^n - P_{k,j}^n) \]

(62)

For \( i = 2 \ldots N_x - 1, j = 2 \ldots N_y - 1 \)

\[ P_{i,j}^{n+1} = P_{i,j}^n + M \left\{ b_0 (P_{i+1,j}^n - P_{i,j}^n) \right\} \]
\[ + \sum_{k=1}^{i-1} (b_k - b_{k-1}) (P_{k+1,j}^n - P_{k,j}^n) + b_0 (P_{i,j+1}^n - P_{i,j}^n) \]
\[ + \sum_{k=1}^{j-1} (b_k - b_{k-1}) (P_{i,k+1}^n - P_{i,k}^n) \]

(63)

For \( i = N_x, j = N_y \)

\[ P_{i,j}^{n+1} = P_{i,j}^n + M \left\{ \sum_{k=1}^{i-1} (b_k - b_{k-1}) (P_{k+1,j}^n - P_{k,j}^n) \right\} \]
\[ + \sum_{k=1}^{j-1} (b_k - b_{k-1}) (P_{i,k+1}^n - P_{i,k}^n) \]

(64)

\[ M = \frac{T_{xy}^n \sigma_{xy}}{A_{i,j}^n} \]
\[ A_{i,j}^n = \left( \frac{V_b \phi c}{\alpha_e B^0 \Delta t} \right)_{i,j} \]

Performing the analysis for point \( i = 2 \) e \( j = 2 \), the error amplification factor is obtained, given by Equation (65).
Inequality (65) is satisfied in the interval given by Equation (66).

\[
0 < M \leq \frac{2(2 - b_1)}{[4(b_1 - 2)^2 + b_1^2]}
\]  

(66)

In the same way as for the transient one-dimensional scheme, the stability of the transient two-dimensional scheme is also a function of the derivative order. Therefore, it is observed that for \( \alpha = 0.9 \), the parameter \( M \) must be within the interval \( 0 < M \leq 0.2592 \) for the numerical scheme to remain stable.

Given the impossibility of obtaining a fixed interval that covers all derivative orders between 0 and 1, numerical experiments are conducted to determine the stability region of the explicit scheme. The procedure for obtaining the stability region of the explicit scheme for the transient two-dimensional model is identical to the procedure used for the transient one-dimensional scheme. Initially, it was observed that the proposed scheme is unstable for all alpha values between 0 and 1, with \( M = 0.5 \). Starting from this point, for the same spatial mesh, the temporal mesh was refined until the explicit scheme became stable.

The results obtained from this numerical experiment are compiled in the graph shown in Figure (4). With reasonable certainty, for \( M \leq 0.25 \) the explicit scheme is stable regardless of the derivative order.

![Figure 4 - Stability region for the explicit solution scheme of the transient two-dimensional model.](image)

Source: authors (2023)
5 CONCLUSIONS

In this work, we presented the modeling of flow in porous media based on the Caputo fractional derivative with respect to spatial coordinate. An explicit numerical scheme was proposed to obtain the pressure profile along a rectangular reservoir for one-dimensional and two-dimensional transient flow. Additionally, a stability analysis of the proposed numerical scheme was conducted using the Von Neumann method, which revealed that the schemes are conditionally stable and dependent on the derivative order ($\alpha$). Finally, the numerical stability regions for the one-dimensional and two-dimensional transient schemes were constructed, showing that the one-dimensional scheme is stable for $M \leq 0.5$ and the two-dimensional scheme is stable for $M \leq 0.25$ for all derivative orders between 0 and 1.

ACKNOWLEDGMENT

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REFERENCES


Numerical Schemes and Stability Analysis for Fractional Models of Transient Uni- and Bidimensional flow in Petroleum Reservoirs


### LIST OF SYMBOLS

**Table 1 - List of symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^\alpha_0 f(t)$</td>
<td>Caputo’s derivative of order $\alpha$ of the function $f(t)$</td>
</tr>
<tr>
<td>$I^\alpha_0 f(t)$</td>
<td>Caputo’s integral of order $\alpha$ of the function $f(t)$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of the fluid</td>
</tr>
<tr>
<td>$u_i$</td>
<td>Flow velocity in the direction “i”</td>
</tr>
<tr>
<td>$A_i$</td>
<td>Perpendicular area to the flow direction “i”</td>
</tr>
<tr>
<td>$\Delta x, \Delta y, \Delta z$</td>
<td>Control volume dimensions</td>
</tr>
<tr>
<td>$q_m$</td>
<td>Mass flow via external source</td>
</tr>
<tr>
<td>$V_b$</td>
<td>Mass flow via external source</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Porosity</td>
</tr>
<tr>
<td>$B$</td>
<td>Factor of volume formation</td>
</tr>
<tr>
<td>$V$</td>
<td>It is the volume in a given condition</td>
</tr>
<tr>
<td>$V_{sc}$</td>
<td>It is the volume under standard conditions</td>
</tr>
<tr>
<td>$\rho_{sc}$</td>
<td>It is the density under standard conditions</td>
</tr>
<tr>
<td>$k_i$</td>
<td>It is the pseudopermeability in the “i” direction</td>
</tr>
<tr>
<td>$\beta_c$</td>
<td>It is a constant of proportionality between the English and metric system of units and is equal to 1.127</td>
</tr>
<tr>
<td>$\mu$</td>
<td>It is the viscosity of the fluid</td>
</tr>
<tr>
<td>$p$</td>
<td>Is the fluid pressure</td>
</tr>
<tr>
<td>$P^n_{ij}$</td>
<td>Is the pressure in a given block $i,j$</td>
</tr>
<tr>
<td>$\partial^\alpha P$</td>
<td>It is the Caputo operator of the fractional derivative of order $\alpha$ of the pressure</td>
</tr>
<tr>
<td>$\bar{\alpha}$</td>
<td>It is the order of the fractional derivative $0 &lt; \alpha &lt; 1$</td>
</tr>
<tr>
<td>$B^0$</td>
<td>Factor of volume formation in standard condition</td>
</tr>
<tr>
<td>$P_0$</td>
<td>It’s the standard pressure</td>
</tr>
<tr>
<td>$c$</td>
<td>Is the compressibility</td>
</tr>
<tr>
<td>$a_c$</td>
<td>It is a proportionality constant for volume between the metric and English system. For units in the English system, $a_c = 5.614583$ and for the metric system, $a_c = 1$.</td>
</tr>
<tr>
<td>$q_{sc}$</td>
<td>It is the mass flux from the external source under standard conditions</td>
</tr>
<tr>
<td>$\Delta x_i$ e $\Delta y_j$</td>
<td>These are the dimensions of the two-dimensional grid block</td>
</tr>
<tr>
<td>$\sigma, b_k$</td>
<td>Parameters of the L1 method of approximation of fractional derivatives</td>
</tr>
<tr>
<td>$M$</td>
<td>Parameter for stability analysis</td>
</tr>
<tr>
<td>$P_0$</td>
<td>Is the initial pressure of the reservoir</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Is the error amplification factor</td>
</tr>
<tr>
<td>$\varepsilon^n$</td>
<td>Is the numerical error at time iteration “n”</td>
</tr>
</tbody>
</table>

**Source:** authors (2023)